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Notes on ordered reproducing Hilbert spaces over the complex plane

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Abstract

Let f, g be entire functions. If there exist $M_1, M_2 > 0$ such that $|f(z)| \leqslant M_1|g(z)|$ whenever $|z| > M_2$ we say that $f \preccurlyeq g$. Let X be a reproducing Hilbert space with an orthogonal basis $\{z^n\}_{n=0}^{\infty}$. We say that X is an ordered reproducing Hilbert space (or X is ordered) if $f \preccurlyeq g$ and $g \in X$ imply $f \in X$. In this note, we show that if $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = \infty$ then X is ordered; if $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = 0$ then X is not ordered. In the case $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = l \neq 0$, there are examples to show that X can be of order or opposite.
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1. Introduction

The simplest example of Hilbert spaces consisting of entire functions that comes to mind is the Fock space or the so called Segal–Bargmann space F. That is the space of all μ-square-integrable entire functions on the complex plane $\mathbb{C}$, where

$$d\mu(z) = \exp\left(-|z|^2\right) dA(z)/2\pi$$

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is the Gaussian measure on $\mathbb{C}$ and dA is the ordinary Lebesgue measure. Note that the Fock space has the following two properties:

(a) the polynomial ring is dense in F;
(b) F has a reproducing kernel $K_\lambda(z) = \exp(z\bar{\lambda})$.

Let $Hol(\mathbb{C})$ denote the ring of all entire functions on the complex plane $\mathbb{C}$ and let X be a Hilbert space contained in $Hol(\mathbb{C})$. We call X a reproducing Hilbert space if X satisfies the above two conditions. Motivated by classifying the quasi-invariant subspace of reproducing Hilbert space, the conception of order relation "$\preccurlyeq$" and ordered reproducing Hilbert space was introduced in [3]. That is, let f and g be entire functions, we say that $f \preccurlyeq g$ if there exist $M_1, M_2 > 0$ such that $|f(z)| \leqslant M_1|g(z)|$ whenever $|z| > M_2$; we say that X is an ordered reproducing Hilbert space (or X is ordered) if $f \preccurlyeq g$ and $g \in X$ implies $f \in X$. The order relation "$\preccurlyeq$" is an useful tool in studying the structure of reproducing Hilbert spaces. For more details, we refer the interest reader to [3,8]. Although some "if and only if" conditions for a space to be ordered are obtained in [3], however, it is never obvious to see which spaces are of order and which spaces are not.

On the other hand, a reproducing Hilbert space is completely determined by its reproducing kernel and if X has an orthogonal basis $\{z^n\}_{n=0}^{\infty}$ then the reproducing kernel of X must have the form $K_\lambda(z) = \sum_{n=0}^{\infty} \gamma_n z^n \bar{\lambda}^n$, where $\gamma_n = 1/\|z^n\|^2$. So it is interesting to know which reproducing spaces are ordered by γ_n, $n = 0, 1, 2, \ldots,$ or equivalently by $\|z^n\|$, $n = 0, 1, 2, \ldots.$

We also note that the ideal by using reproducing kernels to describe the property of the space X (or the property of operators in $B(X)$) is not new. We refer the interested reader to [1,7,10,11] for this aspect and [2,4–6,9,12] for the study of the structure of the reproducing Hilbert spaces.

2. Preliminaries

In this section, we are concerned with the properties of ordered reproducing Hilbert spaces which will be used later. For each polynomial p and f in the Fock space F, by using the Newman–Shapiro isometry theorem [10], there exists a $\delta_p > 0$, such that

$$\|pf\| \geqslant \delta_p \|f\|. \tag{$*$}$$

It is easy to see that the Fock space is ordered. Our first result shows that property $(*)$ in fact characterizes ordered reproducing Hilbert spaces among general reproducing Hilbert spaces.

Proposition 2.1. *Let X be a reproducing Hilbert space on the complex plane $\mathbb{C}$ and $Hol(\mathbb{C})$ the entire function ring. Then the following are equivalent:*

(1) *X is an ordered reproducing Hilbert space;*
(2) *if p is a polynomial and $f \in Hol(\mathbb{C})$, then $pf \in X$ implies $f \in X$;*
(3) *if $\alpha \in \mathbb{C}$ and $f \in Hol(\mathbb{C})$, then $(z+\alpha)f \in X$ implies $f \in X$;*
(4) *for each polynomial p, the multiplication operator M_p is bounded below.*

Proof. (1) $\Rightarrow$ (2). By Proposition 2.5 in [3], $f \preccurlyeq g$ if and only if there exist polynomials p and q with $\deg p \leqslant \deg q$ such that $f/g = p/q$. Thus for each polynomial p, $f \preccurlyeq pf$. Since X is ordered, $pf \in X$ implies $f \in X$.

(2) $\Rightarrow$ (3). Obviously.

(3) $\Rightarrow$ (4). Let $T = M_{z+\alpha}$. Since each polynomial p is of the form $p(z) = (z+\alpha_1)(z+\alpha_2)\cdots(z+\alpha_n)$, we only need to show that the inverse of T is bounded. Obviously, T is injective. We first claim that $R(T)$, the range of T, is closed. In fact, let $[z+\alpha]$ denote the closure of the ideal generated by $z+\alpha$. Then by (3) it is easy to check

$$[z+\alpha] = \{f \in X \mid f(-\alpha) = 0\}.$$

So $R(T) = \{(z+\alpha)f \mid f \in X,\ (z+\alpha)f \in X\} \subseteq [z+\alpha]$. On the other hand, for each $g \in [z+\alpha]$, there exists an entire function g_1 such that $g = (z+\alpha)g_1$. By (3), $g = (z+\alpha)g_1 \in X$ implies $g_1 \in X$. Thus $g \in R(T)$. Thus $R(T) = [z+\alpha]$ is closed.

Next we will show $G(T)$, the graph of T, is closed. That is, for $f_n \xrightarrow{\|\,\|} f$ and $(z+\alpha)f_n \xrightarrow{\|\,\|} g$, we need to show $f \in D(T)$ and $(z+\alpha)f = g$. Since $(z+\alpha)f_n \xrightarrow{\|\,\|} g$. Then $g \in [z+\alpha]$. Thus there exists $g_0 \in X$ such that $g = (z+\alpha)g_0$. Combing this with (3) gives $g_0 \in X$. Since X is an reproducing Hilbert space, $(z+\alpha)f_n \xrightarrow{\|\,\|} (z+\alpha)g_0$ implies $(z+\alpha)f_n \xrightarrow{\text{pointwise}} (z+\alpha)g_0$. Thus $f_n \xrightarrow{\text{pointwise}} g_0$. Since $f_n \xrightarrow{\|\,\|} f$, we have $f = g_0$. Therefore, the graph of T is closed and the inverse of T is bounded.

(4) $\Rightarrow$ (1). Let g be in X, $f \preccurlyeq g$. We only need to show that $f \preccurlyeq g$ implies $f \in X$. In fact, if $f \preccurlyeq g$, then there exist polynomial p and q with $\deg p \leqslant \deg q$ such that $f/g = p/q$, or equivalently, there exists an entire function h such that $f = hp$ and $g = hq$. We assume that $q = (z+\alpha_1)\cdots(z+\alpha_n)$. Then by (4) we have $h \in X$, $(z+\alpha_1)h \in X, \ldots, (z+\alpha_1)\cdots(z+\alpha_n)h \in X$. This meas $z^k h \in X$ for $k = 0, 1, 2, \ldots, n$. Since $\deg p \leqslant \deg q$, we have $pf \in X$. This completes the proof. $\square$

Remark. The equivalence of (1) and (2) was given in [3], we give its proof here just for convenience. The equivalence of (1) and (3) will be used several times in this paper.

Before ending this section, let us observe the properties of reproducing kernels of a class of reproducing Hilbert spaces whose inner product was induced by a positive measure. By the definition of ordered reproducing Hilbert space it is easy to see that such spaces are ordered.

Proposition 2.2. *Let Y be a reproducing Hilbert space. If its inner product is induced by a positive measure, that is $\langle f, g\rangle = \int_{\mathbb{C}} f\bar{g}\,d\nu$, where $f, g \in Y$ and ν is a positive measure, then $\lim_{n\to\infty} \|z^{n+1}\|/\|z^n\| = \infty$.*

Proof. We first prove that if X is a reproducing Hilbert space on $\mathbb{C}$, then

$$\lim_{n\to\infty} \sqrt[n]{\|z^n\|} = \infty.$$

Assume there exists a subsequence $n_1 < n_2 < \cdots$ such that

$$\sqrt[n_k]{\|z^{n_k}\|} \leqslant M$$

for some constant M, that is $\|z^{n_k}\| \leqslant M^{n_k}$. Take $R > M$. Since X is a reproducing Hilbert space, there is a constant C_R such that

$$f(R) \leqslant C_R\|f\| \quad \text{for any } f \in X.$$

This gives

$$R^n \leqslant C_R\|z^n\|.$$

So

$$R \leqslant \sqrt[n_k]{C_R}\left\|z^{n_k}\right\|^{1/n_k} \leqslant M \cdot \sqrt[n_k]{C_R}.$$

Letting $n_k \to \infty$ we get $R \leqslant M$. This contradiction shows $\lim_{n\to\infty} \sqrt[n]{\|z^n\|} = \infty$.

Next we will show that there exists subsequence of $\{\|z^{n_k+1}\|/\|z^{n_k}\|\}_{n=0}^{\infty}$ such that its limit is ∞. Otherwise there exists $M > 0$ such that $\|z^{n+1}\|/\|z^n\| \leqslant M$ for $n = 0, 1, 2, \ldots$. Then

$$\lim_{n\to\infty} \sqrt[n]{\|z^n\|} = \lim_{n\to\infty} \sqrt[n]{\frac{\|z\|}{1} \cdot \frac{\|z^2\|}{\|z\|} \cdots \frac{\|z^n\|}{\|z^{n-1}\|}} \leqslant M.$$

This is a contradiction.

Now to our end we only need to show that $\|z^{n+1}\|/\|z^n\|$ is increasing. In fact,

$$\frac{\|z^{n+1}\|}{\|z^n\|} \Big/ \frac{\|z^n\|}{\|z^{n-1}\|} = \frac{\|z^{n+1}\|\|z^{n-1}\|}{\|z^n\|^2} = \frac{(\int_C |z^{n+1}|^2\, d\sigma)^{1/2}(\int_C |z^{n-1}|^2\, d\sigma)^{1/2}}{\int_C |z^n|^2\, d\sigma}$$
$$\geqslant \frac{\int_C |z^{n+1}||z^{n-1}|\, d\sigma}{\int_C |z^n|^2\, d\sigma} = 1.$$

Thus $\lim_{n\to\infty} \|z^{n+1}\|/\|z^n\| = \infty$. This completes the proof. $\square$

3. Characterization of ordered reproducing Hilbert space

In this section, we are concerned with the relation between the reproducing kernels and the ordered structure of a reproducing Hilbert space. The following is the main result.

Theorem 3.1. *Let X be a reproducing Hilbert space with an orthogonal basis $\{z^n\}_{n=0}^{\infty}$. If $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = \infty$, then X is an ordered reproducing Hilbert space; if $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = 0$, then X is not an ordered reproducing Hilbert space.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{C})$ and $f_N(z) = \sum_{n=0}^{N} a_n z^n$. By (3) of Proposition 1.1, we only need to show that for each complex number α, $(z+\alpha)f \in X$ implies $f \in X$.

Suppose $\alpha = 0$, since $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = \infty$, there exists an integer $N > 0$, such that $\|z^{n+1}\| > 2\|z^n\|$ for $n > N$. Then

$$\left\|f(z)\right\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \left\|z^n\right\|^2 = \sum_{n=0}^{N} |a_n|^2 \left\|z^n\right\|^2 + \sum_{n=N+1}^{\infty} |a_n|^2 \left\|z^n\right\|^2$$
$$\leqslant \sum_{n=0}^{N} |a_n|^2 \left\|z^n\right\|^2 + \sum_{n=N+1}^{\infty} |a_n|^2 \left\|z^{n+1}\right\|^2 \leqslant \sum_{n=0}^{N} |a_n|^2 \left\|z^n\right\|^2 + \|zf\|^2.$$

Thus $zf \in X$ implies $f \in X$.

In the case of $\alpha \neq 0$, by the assumption $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = \infty$, we take $\delta_\alpha > 2|\alpha^2|$ and an integer $N_\alpha > 0$ such that $\|z^{n+1}\|^2/\|z^n\|^2 \geqslant \delta_\alpha$ for $n > N_\alpha$. Therefore for $N > N_\alpha$, we have:

$$\|\alpha f_N\|^2 = \sum_{n=0}^{N} |\alpha|^2 |a_n|^2 \left\|z^n\right\|^2 \leqslant \sum_{n=0}^{N_\alpha} |\alpha|^2 |a_n|^2 \left\|z^n\right\|^2 + \sum_{n=N_\alpha+1}^{N} |\alpha|^2 |a_n|^2 \frac{\|z^{n+1}\|^2}{\delta_\alpha}$$
$$\leqslant M_\alpha + \frac{|\alpha|^2}{\delta_\alpha}\|zf_N\| \leqslant M_\alpha + \frac{1}{2}\|zf_N\|,$$

where $M_\alpha = \sum_{n=0}^{N_\alpha} |\alpha|^2 |a_n|^2 \|z^n\|^2$. Thus

$$\|zf_N\| \leqslant \|(z+\alpha)f_N\| + \|\alpha f_N\| \leqslant \|(z+\alpha)f_N\| + \sqrt{M_\alpha + \frac{1}{2}\|zf_N\|^2}. \tag{1}$$

Since

$$(z+\alpha)f_N(z) = \alpha a_0 - \alpha a_{N+1} z^{N+1} + \sum_{n=1}^{N+1} (a_{n-1} + \alpha a_n) z^n,$$

$$(z+\alpha)f(z) = \alpha a_0 + \sum_{n=1}^{\infty} (a_{n-1} + \alpha a_n) z^n$$

and

$$\big[(z+\alpha)f\big]_N(z) = \alpha a_0 + \sum_{n=1}^{N} (a_{n-1} + \alpha a_n) z^n.$$

Then

$$\begin{aligned} \|(z+\alpha)f_N\| &\leqslant |\alpha a_0| + \big\|\big[(z+\alpha)f\big]_{N+1}\big\| + |\alpha a_{N+1}| \|z^{N+1}\| \\ &\leqslant |\alpha a_0| + \|(z+\alpha)f\| + |\alpha a_{N+1}| \|z^{N+1}\|. \end{aligned} \tag{2}$$

Next we will show that

$$\lim_{n\to\infty} |a_{n+1}| \|z^{n+1}\| \neq \infty \tag{3}$$

Otherwise, there exists $n_0 > 0$ such that $|a_n| \neq 0$ whenever $n > n_0$. Since $(z+\alpha)f(z) = \alpha a_0 + \sum_{n=1}^{\infty} (a_{n-1} + \alpha a_n) z^n \in X$, there exists $n_1 > 0$ such that

$$|a_{n-1} + \alpha a_n| \|z^n\| < 1.$$

So

$$\|\alpha a_n z^n\| - 1 \leqslant \|a_{n-1} z^n\| \leqslant \|\alpha a_n z^n\| + 1, \tag{4}$$

whenever $n > n_1$. Hence if $\lim_{n\to\infty} |a_{n+1}| \|z^{n+1}\| = \infty$ we get

$$\lim_{n\to\infty} \frac{|a_{n-1}|}{|a_n|} = |\alpha|$$

from (4). This contradicts to that f is an entire function. Thus (3) holds. Therefor there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ and $M > 0$ such that

$$\|a_{n_k} z^{n_k}\| \leqslant M \quad \text{for } k = 1, 2, \ldots. \tag{5}$$

If $zf \notin X$, then $\lim_{n\to\infty} \|zf_n\| = \|zf\| = \infty$. Consequently,

$$\lim_{k\to\infty} \|zf_{n_k - 1}\| = \infty.$$

Combining (1), (2) and (5), we get

$$\|zf_{n_k-1}\| \leqslant |\alpha a_0| + \|(z+\alpha)f\| + |\alpha a_{n_k}| \|z^{n_k}\| + \sqrt{M_\alpha + \frac{1}{2}\|zf_{n_k-1}\|^2}. \tag{6}$$

Now divide (6) by $\|zf_{n_k-1}\|$ and let $k \to \infty$, we get $1 \leqslant \sqrt{2}/2$. This is impossible. Thus we have $zf \in X$. So $f \in X$. This completes the proof of the first part.

Next we will show that if $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = 0$, then X is not ordered. In fact, in this case there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\|z^{n_k+1}\|^2/\|z^{n_k}\|^2 < 1/2^k$. Set $g(z) = \sum_{n=0}^{\infty} z^{n_k}/\|z^{n_k}\|$. Since X is an reproducing Hilbert space over the complex plane and $\{z^n\}_{n=0}^{\infty}$ is an orthogonal basis. It is easy to see $g(z)$ is an entire function and $g(z) \notin X$. But,

$$\|zg\|^2 = \sum_{k=0}^{\infty} \frac{\|z^{n_k+1}\|^2}{\|z^{n_k}\|^2} < \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty.$$

So $zg \in X$. Thus, by Proposition 1.1, X is not ordered. This completes the proof. $\square$

The next example is a reproducing Hilbert space X which is not ordered for which $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = \ell \neq 0$.

Example 3.1. Let X be a reproducing Hilbert space with orthogonal basis $\{z^n\}_{n=0}^{\infty}$ and $\|z^n\| = (k+1)^2!$ for $k^2 < n \leqslant (k+1)^2$, $k = 0, 1, 2, \ldots$. Then $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = 1$ and X is not ordered.

Proof. We first set

$$a_k(z) = \frac{z^{(2k)^2+1}}{2k\|z^{(2k)^2+1}\|} + \cdots + \frac{z^{(2k+1)^2-1}}{2k\|z^{(2k+1)^2-1}\|}$$

and let

$$f(z) = \sum_{k=1}^{\infty} a_k(z).$$

It is easy to see that $f(z)$ is an entire function and

$$\|a_k(z)\|^2 = \underbrace{\frac{1}{(2k)^2} + \cdots + \frac{1}{(2k)^2}}_{(2k+1)^2-(2k)^2-1=4k} = \frac{4k}{4k^2} = \frac{1}{k}.$$

Thus $\|f\| = \infty$. So $f \notin X$. But

$$(z-1)f(z) = \sum_{k=1}^{\infty} (z-1)a_k(z) = \sum_{k=1}^{\infty} \left(-\frac{z^{(2k)^2+1}}{2k\|z^{(2k)^2+1}\|} + \frac{z^{(2k+1)^2}}{2k\|z^{(2k+1)^2-1}\|} \right).$$

So

$$\|(z-1)f\|^2 = \sum_{k=1}^{\infty} \left(\frac{1}{(2k)^2} + \frac{1}{(2k)^2} \right) < +\infty$$

and $(z-1)f \in X$. Thus X is not ordered by Proposition 1.1. $\square$

The next example is an ordered reproducing Hilbert space X such that $\liminf_{n\to\infty} \|z^{n+1}\|/\|z^n\| = \ell \neq 0$.

Example 3.2. Let X be a reproducing Hilbert space with orthogonal basis $\{z^n\}_{n=0}^{\infty}$ and $\|z^{2n+1}\|^2 = \|z^{2n}\|^2 = (2n)!$, $n = 0, 1, 2, \ldots$. Then X is ordered.

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$. By Proposition 1.1, we only need to show that $(z+\alpha)f\in X$ implies $f\in X$. It is easy to check that $zf\in X$ implies $f\in X$. In the case of $\alpha\neq 0$,

$$(z+\alpha)f(z)=\alpha a_0+\sum_{n=1}^{\infty}(\alpha a_n+a_{n-1})z^n. \tag{7}$$

We denote the sum of odd terms in the right side of (7) by $f_1(z)$ and the sum of even terms by $f_2(z)$. That is

$$f_1(z)=\alpha a_0+\sum_{k=1}^{\infty}(\alpha a_{2k}+a_{2k-1})z^{2k}, \tag{8}$$

$$f_2(z)=\sum_{k=0}^{\infty}(\alpha a_{2k+1}+a_{2k})z^{2k+1}. \tag{9}$$

Set

$$f_0(z)=\frac{f_2(z)}{z}=\sum_{k=0}^{\infty}(\alpha a_{2k+1}+a_{2k})z^{2k}. \tag{10}$$

Since $(z+\alpha)f(z)\in X$. Then $f_1(z), f_2(z)\in X$. Since $zf_0(z)=f_2(z)\in X$, we have $f_0(z)\in X$ and

$$(f_2-\alpha f_0)(z)=-\alpha^2 a_1+\sum_{k=1}^{\infty}\bigl(a_{2k-1}-\alpha^2 a_{2k+1}\bigr)z^{2k}\in X. \tag{11}$$

So we have

$$\sum_{k=1}^{\infty}\bigl|a_{2k-1}-\alpha^2 a_{2k+1}\bigr|^2\bigl\|z^{2k}\bigr\|^2<+\infty. \tag{12}$$

We also note that Theorem 2.1 in fact shows the following:

Let $\alpha\in\mathbb{C}$ and $g(z)=\sum_{n=0}^{\infty}b_n z^n$ be an entire function and $\lim_{n\to\infty}r_{n+1}/r_n=\infty$, where $r_n>0$. Then $\sum_{n=0}^{\infty}(b_n+\alpha b_{n+1})^2 r_n^2<\infty$ implies $\sum_{n=0}^{\infty}b_n^2 r_n^2<\infty$.

Let $a_{2k+1}=b_k$ and $\|z^{2k}\|=r_k$. Then (12) implies that

$$\sum_{k=0}^{\infty}\bigl\|a_{2k+1}z^{2k}\bigr\|^2<\infty$$

or equivalently

$$h(z)=\sum_{k=0}^{\infty}a_{2k+1}z^{2k}\in X.$$

Since $\|z^{2k+1}\|=\|z^{2k}\|$. We also have $zh(z)=\sum_{k=1}^{\infty}a_{2k+1}z^{2k+1}\in X$ and $f_2(z)-\alpha h(z)=\sum_{k=0}^{\infty}a_{2k}z^{2k+1}\in X$.

By using $\|z^{2k+1}\|=\|z^{2k}\|$ again, we get $h_1(z)=\sum_{k=0}^{\infty}a_{2k}z^{2k}\in X$. Thus $f(z)=h_1(z)+zh(z)\in X$. This completes the proof. $\square$

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